Homogénéisation triple échelle de matériaux composites viscoélastiques

Three scales asymptotic homogenization of viscoelastic composite materials

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Résumé

Une méthode d'homogénéisation asymptotique à plusieurs échelles est proposée pour estimer les propriétés viscoélastiques effectives d'un matériau composite hiérarchique. Les différents niveaux structurels sont analysés sous l'hypothèse d'une périodicité généralisée et se caractérisent par l'utilisation de fonctions stratifiées. Le matériau est modélisé comme un composite viscoélastique linéaire non vieillissant avec une structure hiérarchique en couches. Les problèmes locaux associés, le problème homogénéisé et les expressions des coefficients effectifs sont dérivés pour chaque niveau d'organisation en utilisant le principe de correspondance et la transformation de Laplace-Carson. En travaillant avec une fonction stratifiée générale, des composants anisotropes et un contact parfait aux interfaces, les problèmes locaux et les coefficients effectifs sont calculés analytiquement dans l'espace de Laplace-Carson. L'inversion numérique dans l'espace temporel d'origine est également effectuée. Une approche pour la modélisation du derme en tant que matériau composite viscoélastique hiérarchique est proposée et le calcul du module de relaxation effectif est effectué.

Abstract

A multi-scale asymptotic homogenization method is proposed to estimate the effective viscoelastic properties of a hierarchical composite material. The different structural levels are analyzed under the assumption of a generalized periodicity and are characterized by the use of the stratified functions. The material is modeled as a non-ageing linear viscoelastic composite with a layered hierarchical structure. The associated local problems, the homogenized problem and the expressions of the effective coefficients are derived for each level of organization by using the correspondence principle and the Laplace-Carson transform. Working with a general stratified function, anisotropic components and a perfect contact at the interfaces, the local problems and the effective coefficients are calculated analytically in the Laplace-Carson space. The numerically inversion to the original temporal space is also performed. An approach for modeling the dermis as an hierarchical viscoelastic composite material is proposed and the calculation of the effective relaxation modulus is performed.

Mots Clés : Échelles multiples, Homogénéisation asymptotique, Viscoélasticité linéaire, Derme

Keywords : Multiple scales, Asymptotic homogenization, Linear viscoelasticity, Dermis

1. Introduction

The performance of mechanical properties as weight, heat resistance, corrosion, among others are optimized thanks to the use of composite materials. Specifically, those with viscoelastic properties are widely applied in the aerospace, aeronautical and automobile industry, in bioengineering, as well as in the design of durable and sustainable structural components. Some of these composites, such as biological and synthetic viscoelastic materials, often possess a hierarchical structure and exhibit a multiscale character. The modeling of composite materials requires the development of micromechanics techniques to predict the general (or effective) properties of the heterogeneous structure from the properties, density, proportion and arrangement of its constituents. An excellent review on these methods can be found in [2] and [11]. The existence of a sequence of scales of decreasing order allows to perform a succession of homogenization steps (see [20], [21]). In every step, the homogenization scheme requires the solution of a cell problem with data corresponding to the homogenized material properties of the previous step. On the other hand, many complex heterogeneous structures are characterized by more general periodic functions (see [6]). These functions, called of stratification, describe the microstructure of the composite material and they are related to homogenization problems of shell-type structures of widely applied technological interest (nano-hulls, fiber-reinforced polymers (FRP), civil engineering structures repair, modeling of human heart tissue, among others). Actually, the investigation of the effective properties of non-ageing viscoelastic composites are mainly based on the correspondence principle and the Laplace transform (see [19]). The procedure, see for example [14], consists into the change of the convolution constitutive law, which describes the non-ageing viscoelastic behavior, to a fictitious elastic one in the Laplace domain. Then, the inversion of Laplace transform is considered to derive the effective behavior in the time domain. The present work deals with three-scale Asymptotic Homogenization Method (AHM) which is proposed to modeling a non-ageing linear viscoelastic composite material with generalized periodicity and two hierarchical levels of organization, characterized by the small parameters ε_1 and ε_2 (see Fig. 1). The theoretical aspects of the method are rigorously developed by [9], [1], [5]. The approach aims to generalize the results for elastic composite materials obtained in [3] and to extend them to non-ageing viscoelastic ones with generalized periodicity at each scale of different length. The method is finally applied to investigate the effective viscoelastic properties of the dermis.

2. The linear viscoelastic problem for hierarchical structures

Let us denote by $\Omega \in \mathbb{R}^3$ a linear viscoelastic heterogeneous material possessing two hierarchical levels of organization and a generalized periodicity (see more details of the concept in [6], [7]). Three different scales, namely d_1, d_2 and L, are considered. They characterize the different sizes of the structural levels and are assumed to be well-separated, i.e.

$$\varepsilon_1 = \frac{d_1}{L} << 1 \quad \text{and} \quad \varepsilon_2 = \frac{d_2}{L} << \varepsilon_1. \tag{Eq. 1}$$

At the ε_1-structural level, the domain Ω is occupied by a two-phases quasi-periodic composite such that $\overline{\Omega} = \overline{\Omega}_1^{\varepsilon_1} \cup \overline{\Omega}_2^{\varepsilon_1}$, $\Omega_1^{\varepsilon_1} \cap \Omega_2^{\varepsilon_1} = \emptyset$. We assume that $\Omega_1^{\varepsilon_1} = \cup_{\alpha=1}^{\mathrm{N}_1} {}_{\alpha}\Omega_1^{\varepsilon_1}$ and the interface between $\Omega_1^{\varepsilon_1}$ and $\Omega_2^{\varepsilon_1}$ is denoted by Γ^{ε_1}. Here, Y denotes the unit cell.
At the ε_2-structural level, we consider that for each $\alpha = 1, ..., \mathrm{N}_1$, ${}_{\alpha}\Omega_1^{\varepsilon_1}$ is a two-phase quasi-periodic composite material. Hence, we define, ${}_{\alpha}\overline{\Omega}_1^{\varepsilon_1} = \overline{\Omega}_1^{\varepsilon_2} \cup \overline{\Omega}_2^{\varepsilon_2}$, $\Omega_1^{\varepsilon_2} \cap \Omega_2^{\varepsilon_2} = \emptyset$, $\Omega_1^{\varepsilon_2} = \cup_{\beta=1}^{\mathrm{N}_2} {}_{\beta}\Omega_1^{\varepsilon_2}$ and the interface between $\Omega_1^{\varepsilon_2}$ and $\Omega_2^{\varepsilon_2}$ is denoted by Γ^{ε_2}. In addition, Z is the corresponding unit cell.
Besides, $\boldsymbol{x}\,(x_i)$ represents the global spatial coordinate and two local and independent variables are introduced

$$\boldsymbol{y} = \frac{\boldsymbol{\rho}^{(y)}(\boldsymbol{x})}{\varepsilon_1} \quad \text{and} \quad \boldsymbol{z} = \frac{\boldsymbol{\rho}^{(z)}(\boldsymbol{x})}{\varepsilon_2}, \tag{Eq. 2}$$

The functions $\boldsymbol{\rho}^{(i)} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are the so-called stratified functions. The parametric equations $\boldsymbol{\rho}^{(i)}(\boldsymbol{x}) =$ constant with $\mathrm{i} = \mathrm{y}, \mathrm{z}$ describe the ε_1- and ε_2-structural levels, respectively. Besides, the stratified functions satisfy $\boldsymbol{\rho}^{(i)} \in C^{\infty}(\Omega)$ (see [6]).
Considering that the constitutive response of all the constituents of the composite body is linear viscoelastic, and ignoring inertial terms, the problem in Ω consists in,

$$\begin{aligned}
&\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t) = \boldsymbol{0} && \text{in} && (\Omega \backslash (\Gamma^{\varepsilon_1} \cup \Gamma^{\varepsilon_2})) \times \mathbb{R} \\
&\boldsymbol{u}(\boldsymbol{x},t) = \bar{\boldsymbol{u}} && \text{on} && \partial\Omega_{\mathrm{d}} \times \mathbb{R} \\
&\boldsymbol{\sigma}(\boldsymbol{x},t) \cdot \boldsymbol{n} = \bar{\boldsymbol{S}} && \text{on} && \partial\Omega_{\mathrm{t}} \times \mathbb{R} \\
&\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{0} && \text{in} && \Omega \times \{0\}
\end{aligned} \tag{Eq. 3}$$

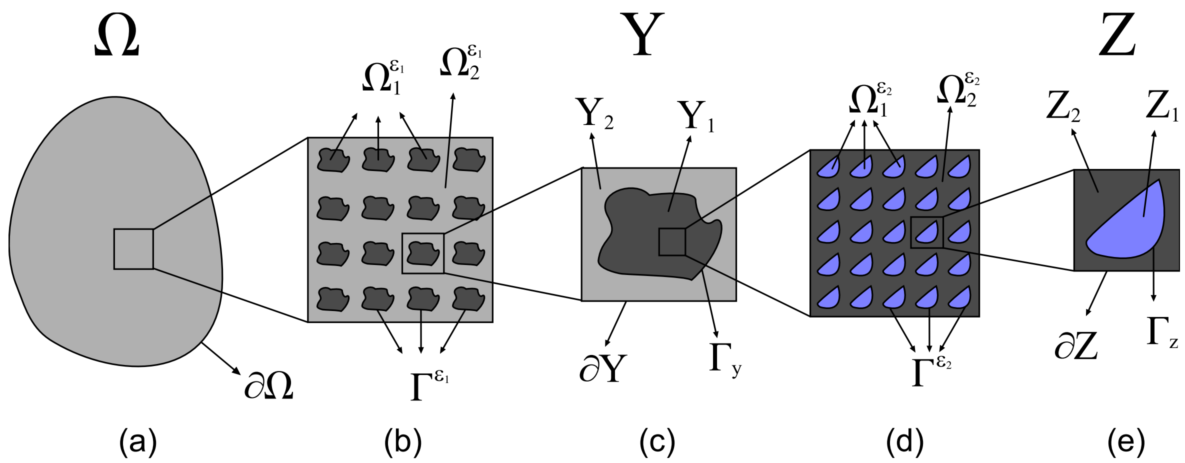

Fig. 1. (a) Macroscale : viscoelastic heterogeneous material. (b) ε_1-structural level. (c) Mesoscale : quasi-periodic cell. (d) ε_2-structural level. (e) Microscale : quasi-periodic cell. The inclusions do not intersect the boundaries.

where $\boldsymbol{\sigma}\,(\sigma_{ij})$ represents the second-order stress tensor, $\boldsymbol{\xi}\,(\xi_{ij})$ denotes the second-order strain tensor, $\boldsymbol{u}\,(u_i)$ is the viscoelastic displacement, $\boldsymbol{f}\,(f_i)$ denotes the action of external volume forces that satisfies $\boldsymbol{f}(\boldsymbol{x},t) \in L^2(\Omega \times \mathbb{R})$. Moreover, $\bar{\boldsymbol{u}}\,(\bar{u}_i)$ and $\bar{\boldsymbol{S}}\,(\bar{S}_i)$ are the prescribed displacement and traction on the boundary $\partial\Omega = \partial\Omega_{\mathrm{d}} \cup \partial\Omega_{\mathrm{n}}$, with $\partial\Omega_{\mathrm{d}} \cap \partial\Omega_{\mathrm{n}} = \emptyset$ and $\boldsymbol{n}\,(n_i)$ is the outward unit vector normal to the surface $\partial\Omega$.
Additionally, the scale-dependent constitutive law which relates the second-order stress and strain tensors is written as follow, (see [15])

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \int_0^t \boldsymbol{\mathscr{R}}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},t-\tau) : \frac{\partial \boldsymbol{\xi}(\boldsymbol{u}(\boldsymbol{x},\tau))}{\partial \tau} d\tau. \qquad \text{(Eq. 4)}$$

Here, the relaxation modulus is denoted by $\boldsymbol{\mathscr{R}}\ (\mathscr{R}_{ijkl})$ and it fulfills the following symmetry properties $\mathscr{R}_{ijkl} = \mathscr{R}_{jikl} = \mathscr{R}_{ijlk} = \mathscr{R}_{klij}$. Also, we assume $\boldsymbol{\mathscr{R}} \in L^\infty(\Omega \times \mathbb{R})$ and that it's positively defined.
Continuity conditions for displacement and traction are imposed on both Γ^{ε_1} and Γ^{ε_2}, i.e.

$$[\![\boldsymbol{u}(\boldsymbol{x},t)]\!] = \boldsymbol{0}, \qquad \left[\!\!\left[\boldsymbol{\sigma}(\boldsymbol{x},t)\cdot \boldsymbol{n}^{(j)}\right]\!\!\right] = \boldsymbol{0}, \qquad (j = y,z) \qquad \text{(Eq. 5)}$$

wherein the outward unit vectors to the surfaces Γ^{ε_1} and Γ^{ε_2} are represented by $\boldsymbol{n}^{(y)} = (n_1^{(y)}, n_2^{(y)}, n_3^{(y)})$ and $\boldsymbol{n}^{(z)} = (n_1^{(z)}, n_2^{(z)}, n_3^{(z)})$, respectively (see Fig. 1). The operator $[\![\cdot]\!]$ denotes the jump across the interface between the two constituents.
In the context of small displacements, the components of the second-order strain tensor $\boldsymbol{\xi}$ are given by the relation

$$\xi_{kl}\,(\boldsymbol{u}\,(\boldsymbol{x},t)) = \frac{1}{2}\left(\frac{\partial u_k\,(\boldsymbol{x},t)}{\partial x_l} + \frac{\partial u_l\,(\boldsymbol{x},t)}{\partial x_k}\right). \qquad \text{(Eq. 6)}$$

The statement of the scale-dependent constitutive law (Eq. 4) corresponds to the special form of non-ageing linear viscoelastic materials (see [17]). Therefore, the problem can be transformed into an elastic one using the Laplace-Carson transform. The aforementioned is known as the correspondence principle. Applying the Laplace-Carson transform to (Eq. 3)-(Eq. 5) and taking into account the summation convention, the mathematical statement for the linear quasi-static viscoelastic heterogeneous problems in the Laplace-Carson space is written,

$$\frac{\partial}{\partial x_j}\left[\mathscr{R}_{ijkl}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)\,\xi_{kl}\,(\boldsymbol{u}\,(\boldsymbol{x},p))\right] + f_i(\boldsymbol{x},p) = 0, \quad \text{in} \quad (\Omega \backslash (\Gamma^{\varepsilon_1} \cup \Gamma^{\varepsilon_2})) \times [0,+\infty). \qquad \text{(Eq. 7)}$$

The corresponding boundary conditions associated to (Eq. 7) are

$$u_i(\boldsymbol{x},p) = \bar{u}_i, \qquad \text{on} \quad \partial\Omega_{\mathrm{d}} \times [0,+\infty), \qquad \text{(Eq. 8)}$$

$$\mathscr{R}_{ijkl}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{x},p)\,\xi_{kl}(\boldsymbol{u}(\boldsymbol{x},p))\,n_j = \overline{S}_i, \quad \text{on} \quad \partial\Omega_{\mathrm{n}} \times [0,+\infty), \qquad \text{(Eq. 9)}$$

and the initial condition is

$$u_i(\boldsymbol{x},p) = 0, \qquad \text{in} \quad \Omega \times \{0\}. \qquad \text{(Eq. 10)}$$

The interface contact conditions become,

$$[\![u_i(\boldsymbol{x},p)]\!] = 0, \quad \left[\!\!\left[\mathscr{R}_{ijkl}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)\,\xi_{kl}(\boldsymbol{u}(\boldsymbol{x},p))\,n_j^{(\mathrm{m})} \right]\!\!\right] = 0 \;\; (m = y,z). \qquad \text{(Eq. 11)}$$

In order to avoid problems with the notation and make the work clearer, from now on, the homogenization process is developed in the Laplace-Carson space and the functions that depend on the parameter p (e.g. $\Phi(p)$) are indicating that we are working on that space.

3. The three-scale asymptotic homogenization method to solve the heterogeneous problem

The aim of this section is to solve the heterogeneous problem (Eq. 7)-(Eq. 11) by using AHM. As the material property is regular in $\boldsymbol{x}$, and periodic in $\boldsymbol{y}$ and $\boldsymbol{z}$ then, according to the chain rule, the derivative in relation to the global coordinate applied to any function in the form $\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ yields the transformation

$$\frac{\partial\Phi_i}{\partial x_j} \to \frac{\partial\Phi_i}{\partial x_j} + \frac{1}{\varepsilon_1}\frac{\partial\rho_l^{(y)}}{\partial x_j}\frac{\partial\Phi_i}{\partial y_l} + \frac{1}{\varepsilon_2}\frac{\partial\rho_m^{(z)}}{\partial x_j}\frac{\partial\Phi_i}{\partial z_m}, \qquad \text{(Eq. 12)}$$

and using a similar idea, the equation (Eq. 6) becomes,

$$\begin{aligned}\xi_{kl}\left(\boldsymbol{\Phi}\left(\boldsymbol{x},\frac{\boldsymbol{\rho}^{(y)}(\boldsymbol{x})}{\varepsilon_1},\frac{\boldsymbol{\rho}^{(z)}(\boldsymbol{x})}{\varepsilon_2}\right)\right) &= \xi_{kl}(\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) + \varepsilon_1^{-1}\xi_{kl}^{y}(\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) \\ &+ \varepsilon_2^{-1}\xi_{kl}^{z}(\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})), \qquad \text{(Eq. 13)}\end{aligned}$$

where

$$\xi_{kl}^{(y)}(\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) = \frac{1}{2}\left(\frac{\partial\rho_m^{(y)}(\boldsymbol{x})}{\partial x_l}\frac{\partial\Phi_k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial y_m} + \frac{\partial\rho_n^{(y)}(\boldsymbol{x})}{\partial x_k}\frac{\partial\Phi_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial y_n}\right), \qquad \text{(Eq. 14)}$$

$$\xi_{kl}^{(z)}(\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) = \frac{1}{2}\left(\frac{\partial\rho_r^{(z)}(\boldsymbol{x})}{\partial x_l}\frac{\partial\Phi_k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial z_r} + \frac{\partial\rho_s^{(z)}(\boldsymbol{x})}{\partial x_k}\frac{\partial\Phi_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial z_s}\right). \qquad \text{(Eq. 15)}$$

3.1. Homogenization procedure

The solution to the problem (Eq. 7)-(Eq. 11) is proposed as follows,

$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x},p) = \tilde{\boldsymbol{u}}^{(0)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p) + \sum_{i=1}^{\infty}\tilde{\boldsymbol{u}}^{(i)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)\varepsilon_2^i, \qquad \text{(Eq. 16)}$$

where $\tilde{\boldsymbol{u}}^{(0)}$ is defined as

$$\tilde{\boldsymbol{u}}^{(0)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p) = \boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p) + \sum_{i=1}^{\infty}\boldsymbol{u}^{(i)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)\varepsilon_1^i. \qquad \text{(Eq. 17)}$$

Here and subsequently (unless necessary), the variable dependence is dropped out for convenience. Then, replacing expansion (Eq. 16) into (Eq. 7) and (Eq. 11), grouping in powers of ε_2 and considering the solvability condition reported in [13], the sequence of problems depicted below can be solved in recursive form. A summary for each problem is proposed.

Problem for ε_2^{-2}

$$\frac{\partial \rho_m^{(z)}}{\partial x_j}\frac{\partial}{\partial z_m}\left[\mathscr{R}_{ijkl}\xi_{kl}^{(z)}\left(\tilde{\boldsymbol{u}}^{(0)}\right)\right]=0, \quad \text{in} \quad (Z\backslash\Gamma_z)\times[0,+\infty), \qquad \text{(Eq. 18)}$$

$$\left[\!\left[\tilde{u}_i^{(0)}\right]\!\right]=0, \qquad \text{on} \quad \Gamma_z\times[0,+\infty), \qquad \text{(Eq. 19)}$$

$$\left[\!\left[\mathscr{R}_{ijkl}\xi_{kl}^{(z)}\left(\tilde{\boldsymbol{u}}^{(0)}\right)n_j^{(z)}\right]\!\right]=0, \qquad \text{on} \quad \Gamma_z\times[0,+\infty). \qquad \text{(Eq. 20)}$$

Since the right hand side of (Eq. 18) is zero, the solvability conditions are satisfied and $\tilde{\boldsymbol{u}}^{(0)}$ is a solution of (Eq. 18) if and only if it is constant in relation to the variable $\boldsymbol{z}$, i.e.

$$\tilde{\boldsymbol{u}}^{(0)}=\tilde{\boldsymbol{u}}^{(0)}(\boldsymbol{x},\boldsymbol{y},p) \qquad \text{(Eq. 21)}$$

and taking into account (Eq. 17), it is obtain

$$\boldsymbol{u}^{(0)}=\boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},p),$$

$$\boldsymbol{u}^{(i)}=\boldsymbol{u}^{(i)}(\boldsymbol{x},\boldsymbol{y},p).$$

Problem for ε_2^{-1}

Using (Eq. 21) and the fact that $\xi_{kl}^{(z)}\left(\tilde{\boldsymbol{u}}^{(0)}\right)=0$, the problem leads to

$$\frac{\partial \rho_m^{(z)}}{\partial x_j}\frac{\partial}{\partial z_m}\left[\mathscr{R}_{ijkl}\left(\xi_{kl}^{(z)}\left(\tilde{\boldsymbol{u}}^{(1)}\right)+\xi_{kl}\left(\tilde{\boldsymbol{u}}^{(0)}\right)+\varepsilon_1^{-1}\xi_{kl}^{(y)}\left(\tilde{\boldsymbol{u}}^{(0)}\right)\right)\right]=0 \qquad \text{(Eq. 22)}$$
$$\text{in } Z\backslash\Gamma_z\times[0,+\infty)$$

$$\left[\!\left[\tilde{u}_i^{(1)}\right]\!\right]=0, \qquad \text{(Eq. 23)}$$

$$\left[\!\left[\mathscr{R}_{ijkl}\left[\xi_{kl}^{(z)}\left(\tilde{\boldsymbol{u}}^{(1)}\right)+\xi_{kl}\left(\tilde{\boldsymbol{u}}^{(0)}\right)+\varepsilon_1^{-1}\xi_{kl}^{(y)}\left(\tilde{\boldsymbol{u}}^{(0)}\right)\right]n_j^{(z)}\right]\!\right]=0, \qquad \text{(Eq. 24)}$$
$$\text{on } \Gamma_z\times[0,+\infty).$$

The divergence theorem and the z-periodicity condition of $\mathscr{R}_{ijkl}$ ensure the fulfillment of the solvability conditions in (Eq. 22). Thus, the existence and uniqueness of a solution for the problem (Eq. 22)-(Eq. 24) is guaranteed (see [13]). Applying separation of variables, a general solution for (Eq. 22)-(Eq. 24) is given by

$$\tilde{u}_m^{(1)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)=\tilde{\chi}_{klm}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},p)\,\tilde{U}_{kl}^{(0)}(\boldsymbol{x},\boldsymbol{y},p), \qquad \text{(Eq. 25)}$$

where

$$\tilde{U}_{kl}^{(0)}(\boldsymbol{x},\boldsymbol{y},p)=\xi_{kl}\left(\tilde{\boldsymbol{u}}^{(0)}(\boldsymbol{x},\boldsymbol{y},p)\right)+\varepsilon_1^{-1}\xi_{kl}^{(y)}\left(\tilde{\boldsymbol{u}}^{(0)}(\boldsymbol{x},\boldsymbol{y},p)\right). \qquad \text{(Eq. 26)}$$

The third rank tensor $\tilde{\chi}_{klm}$ is y- and z-periodic. Then, substituting (Eq. 25) into (Eq. 22)-(Eq. 24) and after some simplifications the following auxiliary problem, which we call the ε_2-cell problem, is obtained

$$\frac{\partial \rho_m^{(z)}}{\partial x_j}\frac{\partial}{\partial z_m}\left[\mathscr{R}_{ijkl}+\mathscr{R}_{ijpq}\xi_{pq}^{(z)}\left(\tilde{\boldsymbol{\chi}}_{kl}\right)\right]=0, \quad \text{in } (Z\backslash\Gamma_z)\times[0,+\infty), \qquad \text{(Eq. 27)}$$

$$[\tilde{\chi}_{klm}]=0, \qquad \text{on} \quad \Gamma_z\times[0,+\infty), \qquad \text{(Eq. 28)}$$

$$\left[\!\left[\left(\mathscr{R}_{ijkl}+\mathscr{R}_{ijpq}\xi_{pq}^{(z)}\left(\tilde{\boldsymbol{\chi}}_{kl}\right)\right)n_j^{(z)}\right]\!\right]=0, \qquad \text{on} \quad \Gamma_z\times[0,+\infty). \qquad \text{(Eq. 29)}$$

Before proceeding, let's introduce first the following notation. Since the quantities involved vary on the y and z scales, the following cell average operators are defined

$$\langle\bullet\rangle_y = \frac{1}{|\mathrm{Y}|}\int_{\mathrm{Y}}(\bullet)dy \quad \text{and} \quad \langle\bullet\rangle_z = \frac{1}{|\mathrm{Z}|}\int_{\mathrm{Z}}(\bullet)\mathrm{d}z, \qquad \text{(Eq. 30)}$$

where $|\mathrm{Y}|$ and $|\mathrm{Z}|$ represent the periodic cell volumes.

Problem for ε_2^0

The expressions are grouped in powers of ε_2^0. Applying the average operator $\langle\bullet\rangle_z$, taking into account the divergence theorem and the z-periodicity of the involved functions, we obtain

$$\left(\frac{\partial}{\partial x_j} + \varepsilon_1^{-1}\frac{\partial \rho_n^{(\mathrm{y})}}{\partial x_j}\frac{\partial}{\partial y_n}\right)\breve{\mathscr{R}}_{ijkl}\tilde{U}_{kl}^{(0)} + f_i = 0 \quad \text{in} \quad \Omega_1^{\varepsilon_1}\times[0,+\infty), \qquad \text{(Eq. 31)}$$

where

$$\breve{\mathscr{R}}_{ijkl} = \left\langle \mathscr{R}_{ijkl} + \mathscr{R}_{ijpq}\xi_{pq}^{(z)}\left(\tilde{\boldsymbol{\chi}}_{kl}\right)\right\rangle_z \qquad \text{(Eq. 32)}$$

is the effective coefficient at the ε_1-structural level. Note that $\breve{\mathscr{R}}_{ijkl} = \breve{\mathscr{R}}_{ijkl}(\boldsymbol{x},\boldsymbol{y},p)$.

We have carried out the analysis for the ε_1-structural level. In order to find the effective behaviour of the hierarchical composite, a similar procedure is performed. Then, using relations (Eq. 17) and (Eq. 26) into equation (Eq. 31) and grouping in powers of ε_1, the following theoretical results are obtained,

Problem for ε_1^{-2}

The solvability condition is also satisfied and the following result is reached

$$\boldsymbol{u}^{(0)} = \boldsymbol{u}^{(0)}(\boldsymbol{x},p). \qquad \text{(Eq. 33)}$$

Problem for ε_1^{-1}

A general solution for the problem is given by

$$u_m^{(1)}(\boldsymbol{x},\boldsymbol{y},p) = \chi_{klm}(\boldsymbol{x},\boldsymbol{y},p)\,\xi_{kl}\left(\boldsymbol{u}^{(0)}(\boldsymbol{x},p)\right) \qquad \text{(Eq. 34)}$$

where the third order tensor χ_{klm} is y-periodic and is the solution of the following problem, referred to as the ε_1-cell problem,

$$\frac{\partial \rho_m^{(\mathrm{y})}}{\partial x_j}\frac{\partial}{\partial y_m}\left[\breve{\mathscr{R}}_{ijkl} + \breve{\mathscr{R}}_{ijpq}\xi_{pq}^{(y)}\left(\boldsymbol{\chi}_{kl}\right)\right] = 0 \quad \text{in} \quad Y\backslash\Gamma_{\mathrm{Y}}\times[0,+\infty), \qquad \text{(Eq. 35)}$$

$$[\![\chi_{klm}]\!] = 0 \quad \text{on} \quad \Gamma_{\mathrm{Y}}\times[0,+\infty), \qquad \text{(Eq. 36)}$$

$$\left[\!\!\left[\left(\breve{\mathscr{R}}_{ijkl} + \breve{\mathscr{R}}_{ijpq}\xi_{pq}^{(y)}\left(\boldsymbol{\chi}_{kl}\right)\right)n_j^{(\mathrm{y})}\right]\!\!\right] = 0 \quad \text{on} \quad \Gamma_{\mathrm{Y}}\times[0,+\infty). \qquad \text{(Eq. 37)}$$

Problem for ε_1^0

The *homogenized problem* is obtained and it can be written in the form

$$\frac{\partial}{\partial x_j}\left[\hat{\mathscr{R}}_{ijkl}\xi_{kl}\left(\boldsymbol{u}^{(0)}\right)\right] + f_i = 0 \quad \text{in} \quad \Omega\times[0,+\infty) \qquad \text{(Eq. 38)}$$

where

$$\hat{\mathscr{R}}_{ijkl} = \left\langle \breve{\mathscr{R}}_{ijkl} + \breve{\mathscr{R}}_{ijpq}\xi_{pq}^{(y)}\left(\boldsymbol{\chi}_{kl}\right)\right\rangle_y. \qquad \text{(Eq. 39)}$$

is the expressions for the effective relaxation modulus of the hierarchical composite material. Finally, from (Eq. 8)-(Eq. 10) and applying the cell average operators over Z and Y, the boundary conditions for (Eq. 38) are

$$\hat{\mathscr{R}}_{ijkl}\xi_{kl}\left(\boldsymbol{u}^{(0)}\right)n_j = \overline{S}_i \qquad \text{on} \quad \partial\Omega_{\mathrm{n}} \times [0,+\infty), \qquad \text{(Eq. 40)}$$

$$u_i^{(0)} = \overline{u}_i \qquad \text{on} \quad \partial\Omega_d \times [0,+\infty), \qquad \text{(Eq. 41)}$$

and the initial condition

$$u_i^{(0)} = 0 \qquad \text{in} \quad \Omega \times \{0\}. \qquad \text{(Eq. 42)}$$

4. Effective coefficients for hierarchical laminated composites with generalized periodicity

Laminated composites materials can be described by using stratified function in the form $\rho : \mathbb{R}^n \to \mathbb{R}$ with $n = 2,3$ (see [6]). Now, we consider the general case when the stratified functions are $\rho^{(i)} : \mathbb{R}^3 \to \mathbb{R}$ with $i = y,z$. Hence, following the same procedure carried out by [4], the local problems (Eq. 27)-(Eq. 29) and (Eq. 35)-(Eq. 37) and the effective coefficients (Eq. 32) and (Eq. 39) are transformed using the Voigt's notation and can be solved analytically in recursive form.
In the process, we integrate each equation in relation to the local variables and determining the constants of integration by using the corresponding cell average operator. It is really worthy to remark that the components of the local problems are raised for the anisotropic case which is an advantage of this approach.

5. An approach for modeling the mechanical properties of the dermis

Nowadays, many researches are focus on the study of the components and the structure of the skin. A better understanding of this soft biological tissue has a real impact on biomedical applications and also inspires modern technology such as flexible electronics, soft robotics and prosthetics (see [18]).
Skin has three main layers : the epidermis, the dermis and the hypodermis. The most important mechanical and thermal unit of the skin is the dermis ; it represents the 90% of the thickness of the skin (see [12]).
Also, the dermis is considered a multilayer collagen-reinforced structure. Two of their principal components are the collagen and the ground substance (supporting matrix). The collagen contributes to 75% of the fat-free dry mass and 18%–30% of the volume of dermis and can be considered like a linear elastic material. On the other hand, the ground substance is responsible for the viscoelastic behavior of the dermis and comprises about 20% of the dry weight of skin and makes up between 70% and 90% of the skin's volume. It is a gel like substance containing a class of chemicals including glycosaminoglycans(GAG), proteoglycans and glycoproteins (see [8])
In this section, an approach for modeling the dermis as an hierarchical viscoelastic composite material is proposed. We follow the structural ideas depicted in the Fig 3 of [18]. It is stated that the stiffness of collagenous materials, which are within the ground substance [10] and form the dermis, is a consequence of the properties, arrangement, and geometric distribution of collagen fibrils. These structures are long strands with wavy effects and are arranged forming wavy parallel fibers, which are flattened in the plane of the dermis and determine its properties. We present the problematic from the point of view of laminated materials (see Fig 2).
Mechanical properties of materials can be found in Tab. 1 (see [16]). Besides, the viscoelastic constituent (ground substance) can be modeled using normalized Prony series,

$$G(t) = G_\infty + \sum_{n=1}^{N} G_n e^{(-t/\tau_n)}, \qquad \text{(Eq. 43)}$$

where τ_n are the relaxation times, G_n are the modulus coefficients, $= G_\infty$ is the long-term modulus (Tab. 2). For sake of simplicity in the model, only two term in the Prony series (see [10]) are considered.

	Young modulus (GPa)	Poisson ratio
Collagen	1	0.48
Ground of substance	0.0102	0.48

Tab. 1. Mechanical properties for the constituents of the dermis.

	G_1	G_2	τ_1	τ_1
Ground of substance	0.22	0.28	0.31	6.96

Tab. 2. Coefficients of the Prony series.

The average operator is calculated

$$\langle f \rangle = V_1 f_{(1)} + V_2 f_{(2)}, \qquad \text{(Eq. 44)}$$

where the subscripts (1), (2) are indicating the corresponding material and V_i represents the volume fractions of each constituent.

The outcomes in the calculation of the effective relaxation modulus are displayed in Fig 3. The methodology allows to estimate the effective behavior for the dermis.

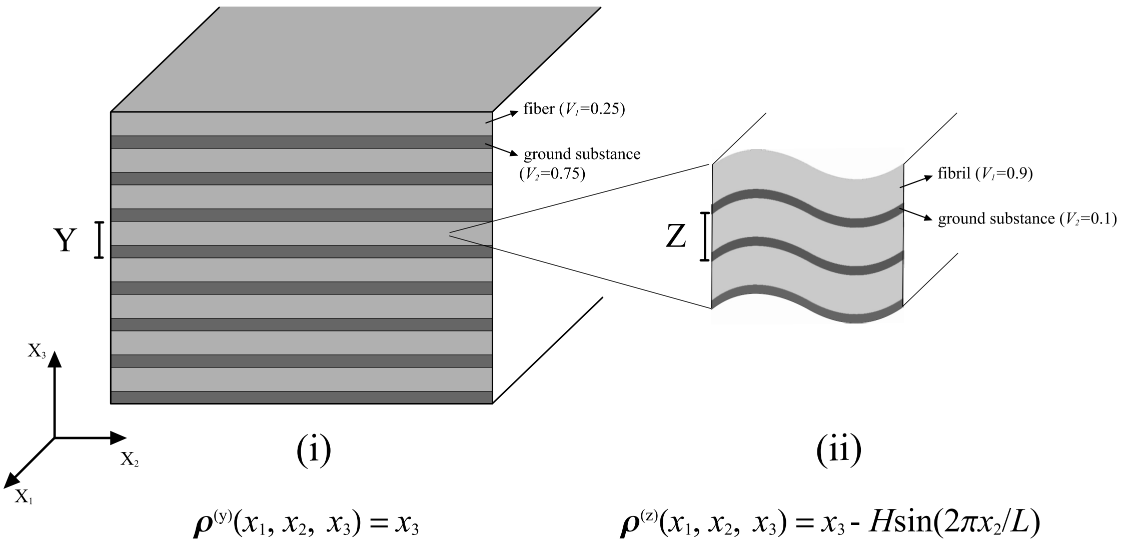

Fig. 2. The hierarchical structure of the dermis.

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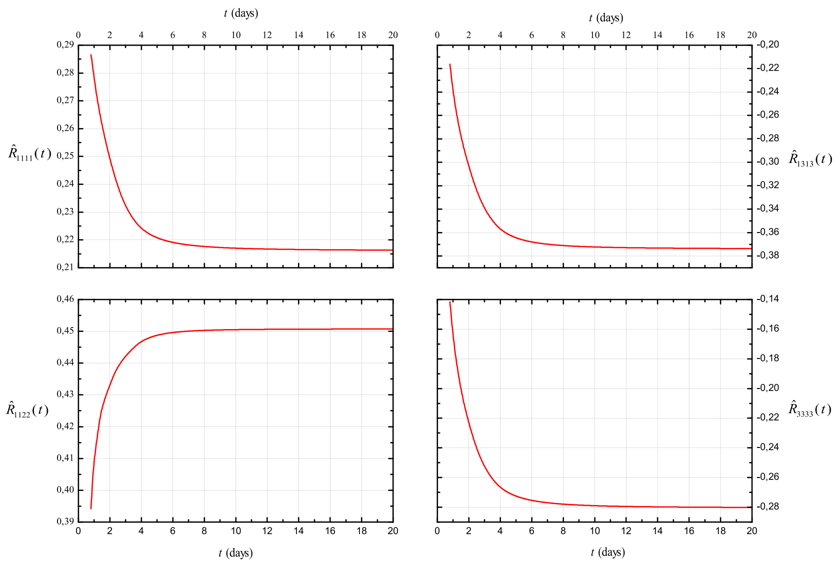

Fig. 3. Computation of the effective relaxation modulus for the dermis. $H/L = 0.25$

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